2024-2025
Fall Semester

Course of Power Systems Analysis

Fundamental aspects for the study of AC circuits

Prof. Mario Paolone
Distributed Electrical Systems Laboratory
École Polytechnique Fédérale de Lausanne (Switzerland)

Outline

Introduction

Single-phase AC circuits

Powers in AC circuits

Three-phase AC circuits

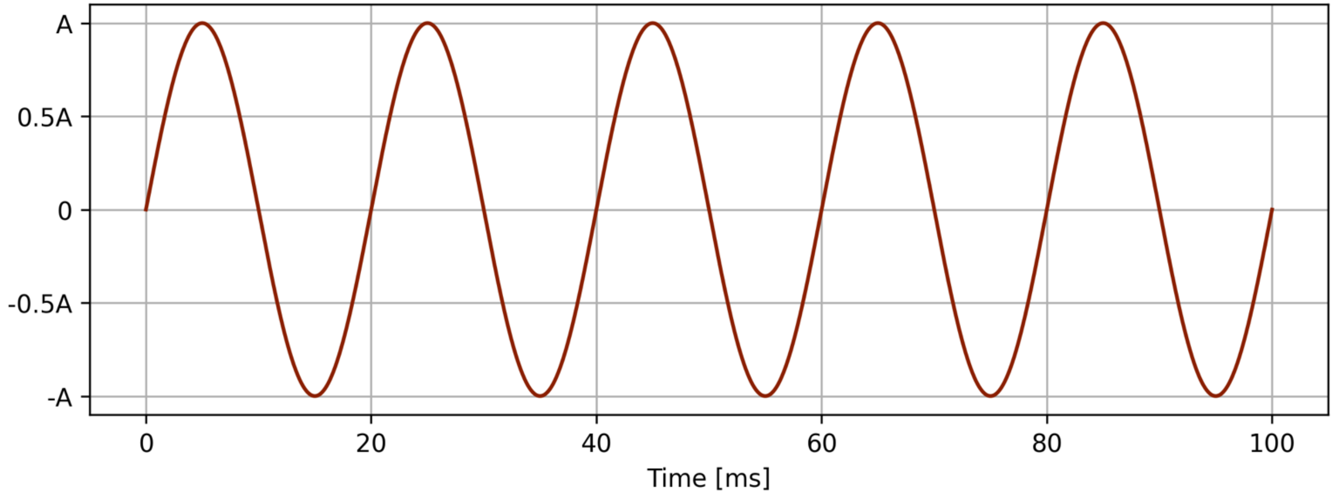
Introduction

Waveforms in the time domain

The waveform of generic power systems's quantities (e.g. a bus voltage or a line current) can be assumed to be **purely sinusoidal and of constant frequency**.

$$a(t) = A_{max} \sin(\omega t + \theta)$$

$A_{max} \in \mathbb{R}^+$	amplitude: max value of $a(t)$.
$\omega \in \mathbb{R}^+$	angular frequency [1/s]
$\theta \in \mathbb{R}$	phase angle [rad]



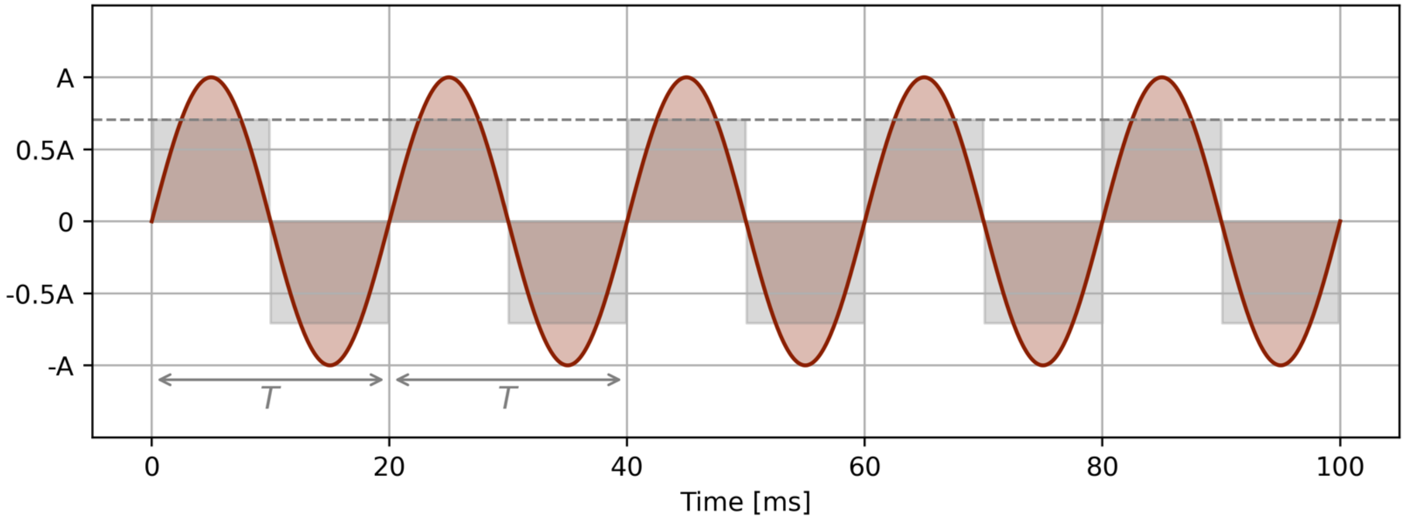
Introduction

Waveforms in the time domain

$$a(t) = A_{max}\sin(\omega t + \theta)$$

the period, in [s], of the waveform is defined as $T = 2\pi/\omega$, and its frequency, in [Hz], as $f = 1/T = \omega/2\pi$.
Finally, the Root Mean Square (RMS) is:

$$A = \sqrt{\frac{1}{T}\int_t^{t+T} A_{max}^2 \cos^2(\omega t + \theta)dt} = \frac{A_{max}}{\sqrt{2}} \cong 0.707 A_{max}$$



Introduction

Iso-frequency quantities

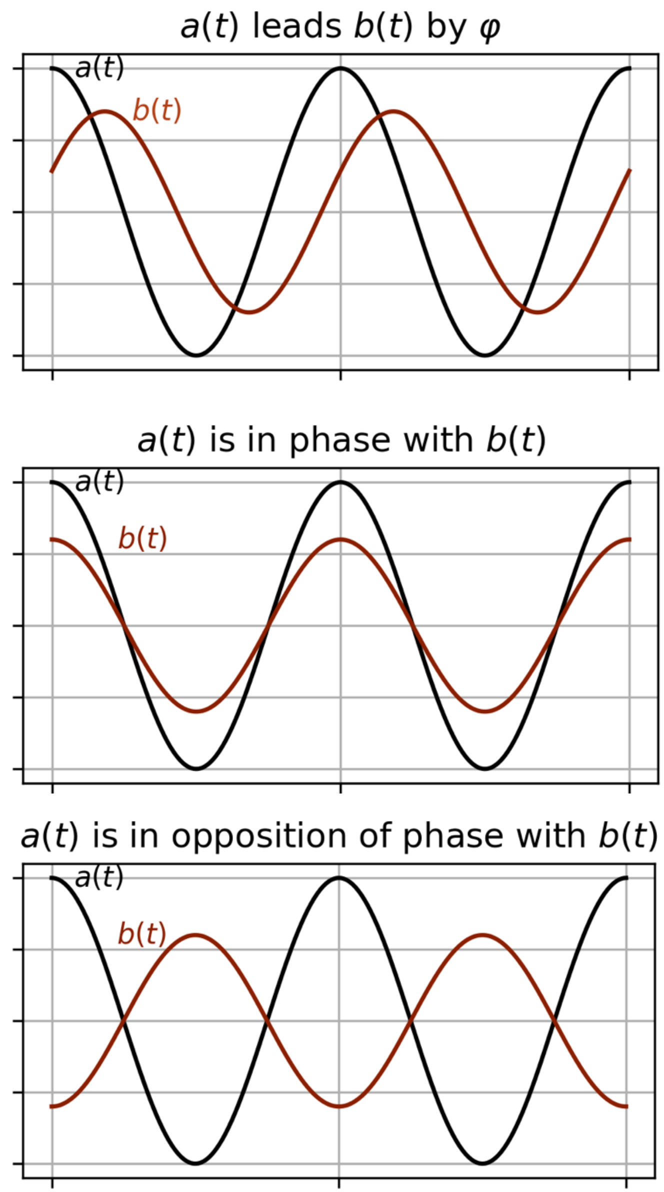
$$a(t) = \sqrt{2}\, A \sin(\omega t + \theta_a)$$
$$b(t) = \sqrt{2}\, B \sin(\omega t + \theta_b)$$

Note that $a(t)$ and $b(t)$ have the same angular frequency. The phase angle shift between $a(t)$ and $b(t)$ is:

$$\varphi = \theta_a - \theta_b$$

With:

$\varphi = 0$	$a(t)$ and $b(t)$ are in phase.
$\varphi > 0$	$a(t)$ leads $b(t)$ by φ.
$\varphi < 0$	$a(t)$ lags $b(t)$ by φ.



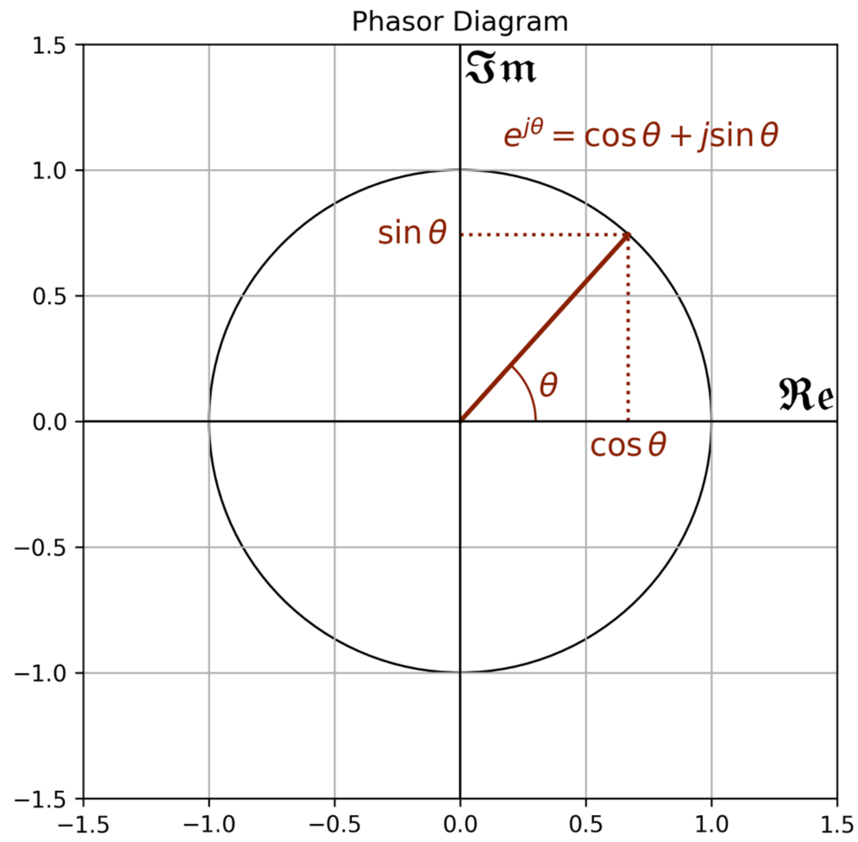
Introduction

Euler identity

For any real number θ:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

where the inputs of the trigonometric functions $\sin$ and $\cos$ are given in radians.



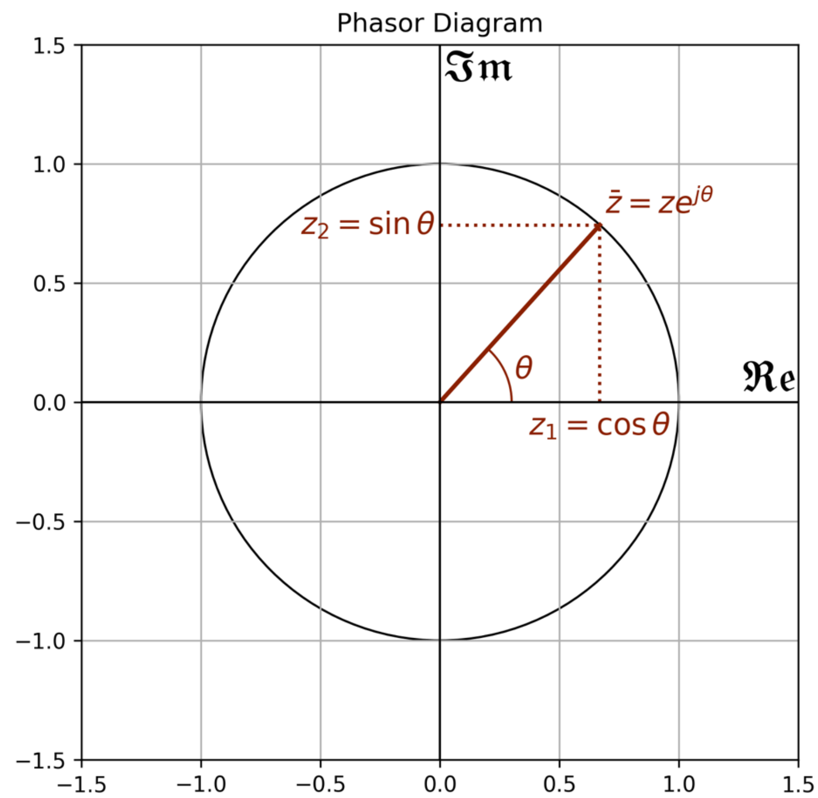
Introduction

Euler identity

Geometric interpretation:

Any complex number $\overline{z} = z_1 + jz_2$ can be represented in polar coordinates as (z, θ_z) , where:

- $z = |\overline{z}|$ is the distance from the origin
- $\theta = \angle\overline{z}$ is the angle counterclockwise from the positive x-axis).



According to Euler's identify, this is equivalent to saying

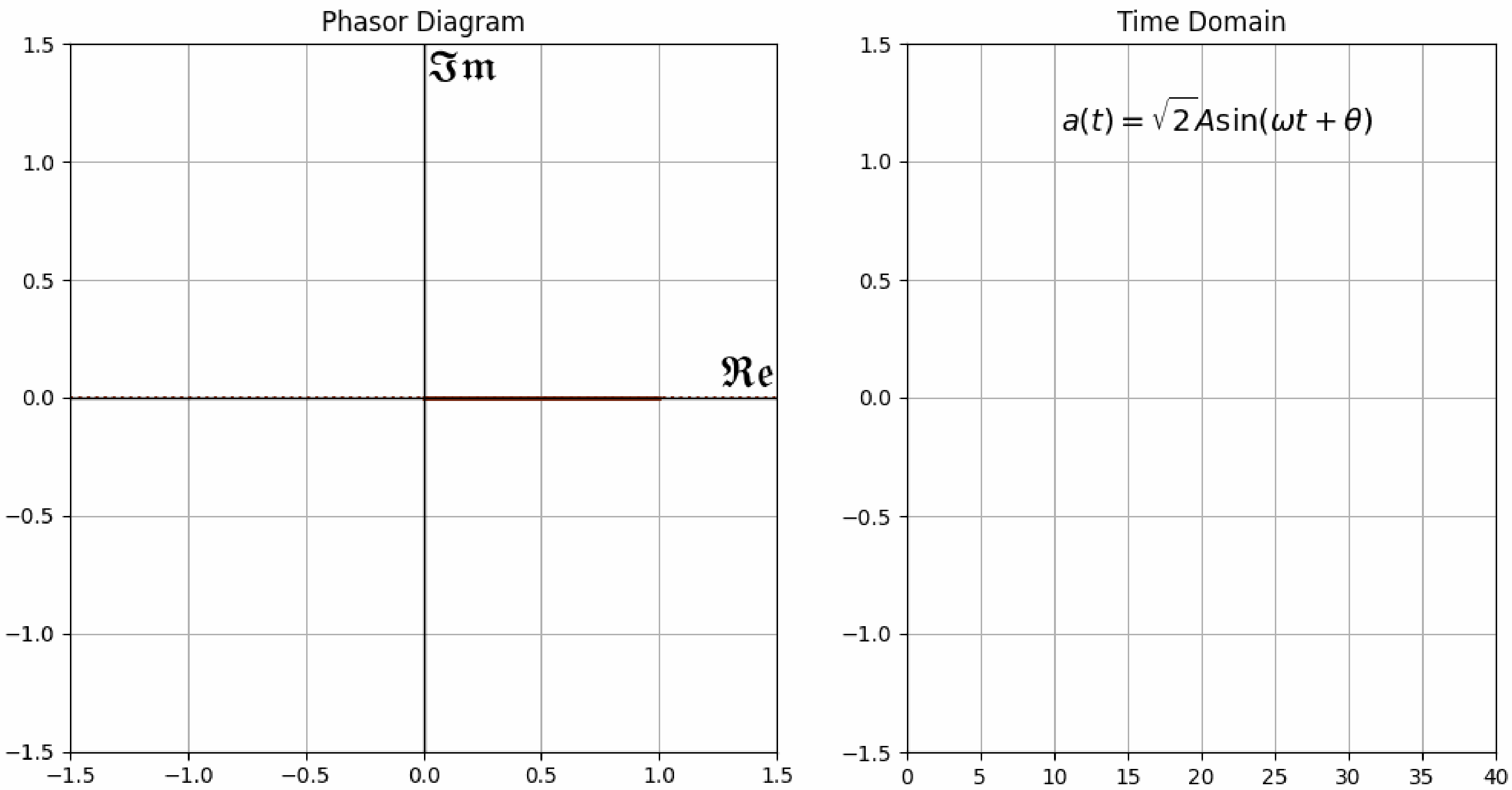
$$\overline{z} = ze^{j\theta}$$

Introduction

Sinusoids and phasors

$$a(t) = \sqrt{2}A\cos(\omega t + \theta) = \Re\mathrm{e}\left[\sqrt{2}Ae^{j(\omega t+\theta)}\right] = \Re\mathrm{e}\left[\sqrt{2}Ae^{j\theta}e^{j\omega t}\right] = \Re\mathrm{e}\left[\sqrt{2}\,\overline{A}e^{j\omega t}\right]$$

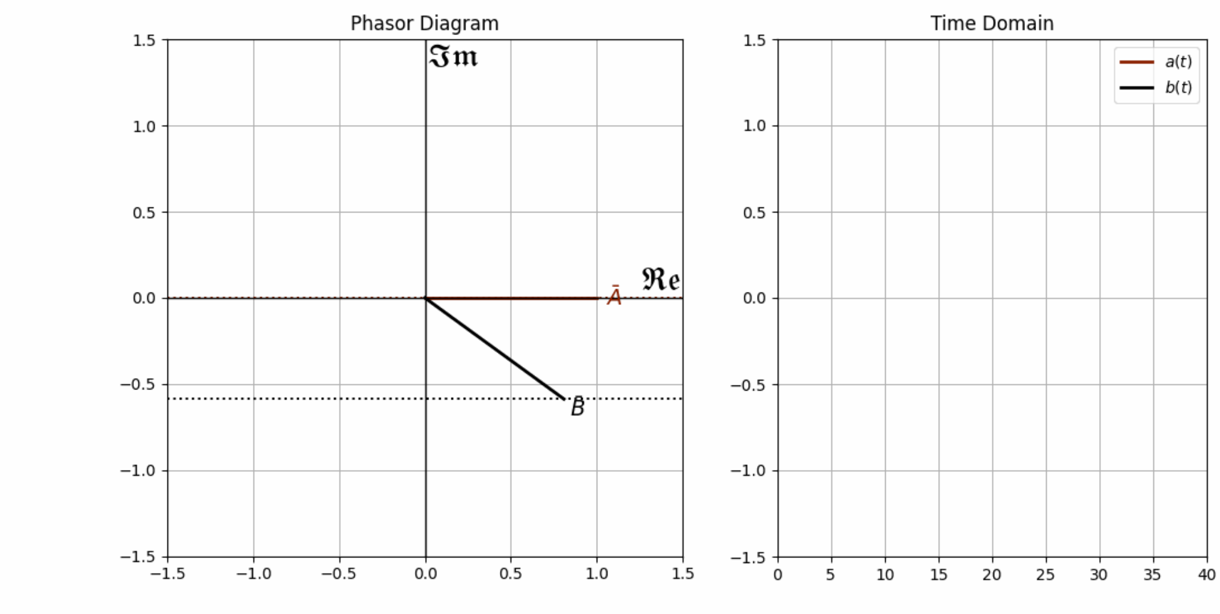
$$a(t) = \sqrt{2}A\sin(\omega t + \theta) = \Im\mathrm{m}\left[\sqrt{2}Ae^{j(\omega t+\theta)}\right] = \Im\mathrm{m}\left[\sqrt{2}Ae^{j\theta}e^{j\omega t}\right] = \Im\mathrm{m}\left[\sqrt{2}\,\overline{A}e^{j\omega t}\right]$$



Introduction

Sinusoids and phasors

$$a(t) = A_{max}\sin(\omega t + \theta) = \mathfrak{Im}\left[\sqrt{2}\,Ae^{j(\omega t+\theta)}\right] = \mathfrak{Im}\left[\sqrt{2}\,Ae^{j\theta}e^{j\omega t}\right] = \mathfrak{Im}\left[\sqrt{2}\,\bar{A}e^{j\omega t}\right]$$



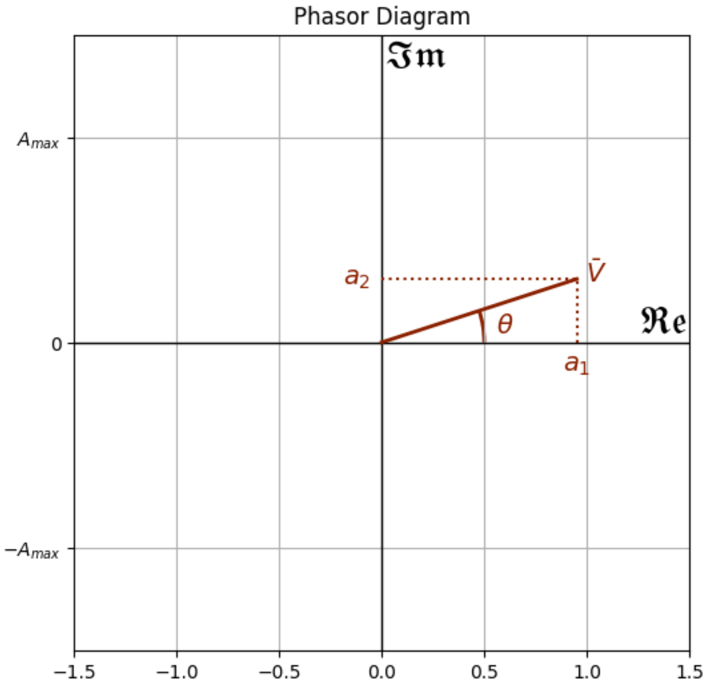
Where the phasor is a current or a voltage given by:

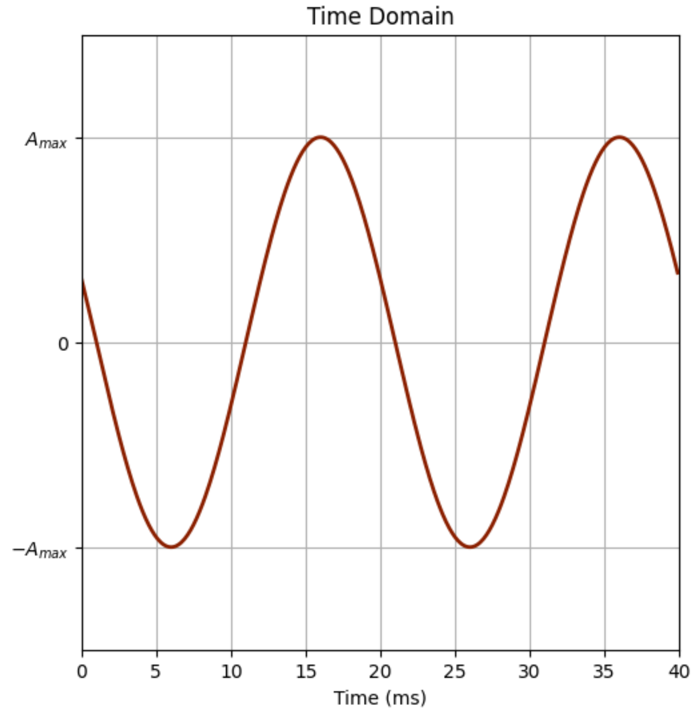
$$\bar{A} = Ae^{j\theta} = A(\cos\theta + j\sin\theta) = A\angle\theta$$

In other words, we have a bijective transformation between phasors and time-domain sinusoidal quantities.

Introduction

Sinusoids and phasors





Phasor domain

Time domain

$$\bar{A} = A(\cos\theta + j\sin\theta)$$
$$\bar{A} = Ae^{j\theta}$$
$$\bar{A} = A\angle\theta$$

$$a(t) = \sqrt{2}A\sin(\omega t + \theta)$$

Introduction

Complex plane

Geometric interpretation

$$\overline{A} = Ae^{j\theta} = A(\cos\theta + j\sin\theta) = a_1 + ja_2$$

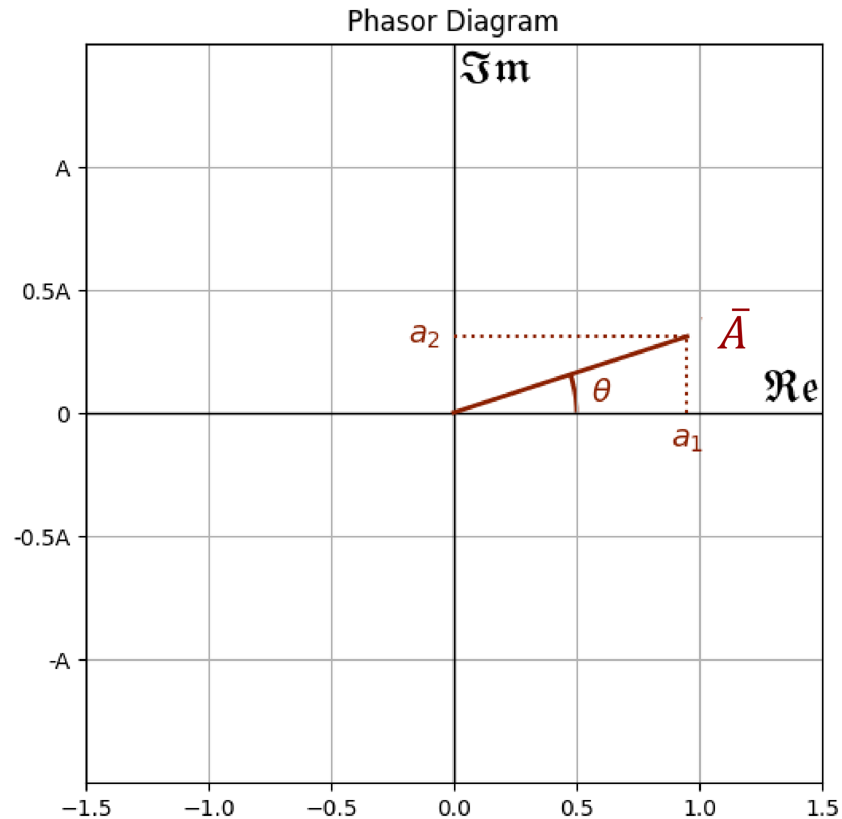
where:

$$a_1 = A\cos\theta$$

$$a_2 = A\sin\theta$$

$$A = \sqrt{a_1^2 + a_2^2}$$

$$\theta = \tan^{-1}\frac{b}{a} \text{ where } a > 0$$



Introduction

Phasors properties

- **Uniqueness**:
  Two sinusoids at the same frequency are equal if and only if they are represented by the same phasor:
  $$a(t) = b(t) \Leftrightarrow \overline{A} = \overline{B}$$

- **Linearity**:
  The linear combination of phasors represents the same linear combination of sinusoids at the same frequency
  $$c_1 a(t) + c_2 b(t) \Leftrightarrow c_1 \overline{A} + c_2 \overline{B}, c_1, c_2 \in \mathbb{R}$$

- **Derivative**:
  If $\overline{A}$ is the phasor of $a(t)$, the time derivative of $a(t)$ is given by $j\omega\overline{A}$.

Outline

Introduction

Single-phase AC circuits

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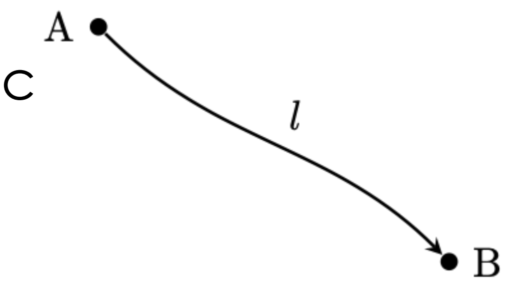
Three-phase AC circuits

Single phase AC circuits

Electric voltage an potential difference

Voltage, or **electric potential difference**, between two points A and B say V_{AB}, associated to an electric field $\mathbf{E}$ is defined as the work done by an external force to move a unit positive charge (i.e., of $1\ C$) from point A to point B without any acceleration:



$$V_{AB} = \int_{A,l}^{B} \mathbf{E} \cdot d\boldsymbol{l}$$

where:

- $\mathbf{E}$ is the electric field.
- $d\boldsymbol{l}$ is an infinitesimal vector element of the path from A to B.

In **electrostatics** (and **slowly time-varying phenomena**), the electric field is **conservative** for which we have that:

$$\oint_C \mathbf{E} \cdot d\boldsymbol{l} = 0$$

This implies that V_{AB} is **path-independent**, therefore: $V_{AB} = V_A - V_B$

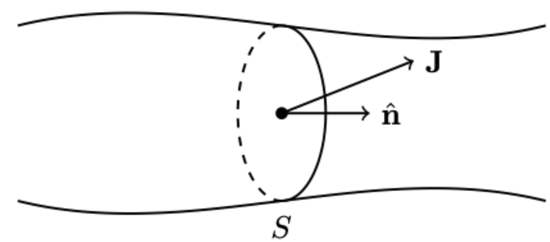
Single phase AC circuits

Electric current

Electric **current** i is defined as the rate at which charge q passes through an oriented surface S (see figure). Considering that the **punctual charge flow rate defines** the **current density** vector $\mathbf{J}$, we get:

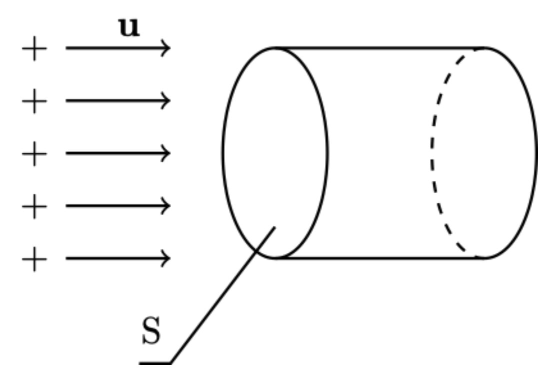
$$i = \iint_S \mathbf{J} \cdot \hat{\boldsymbol{n}} dS = \frac{dq}{dt}$$

where $\hat{\boldsymbol{n}}$ is the unity vector perpendicular to a generic point on surface S.



For a surface S perpendicular to the current density vector, assuming $\mathbf{J}$ uniform across S, we have:

$$i = \rho_C u S = |\mathbf{J}| S$$



where ρ_C is the charge density and u is the velocity of the charge carriers.

Single phase AC circuits

Circuit analysis

Kirchhoff's Current Law (KCL) states that the sum of currents entering/leaving a node that has n incident conductors, is zero. KCL is a simple extension of the charge conservation principle.

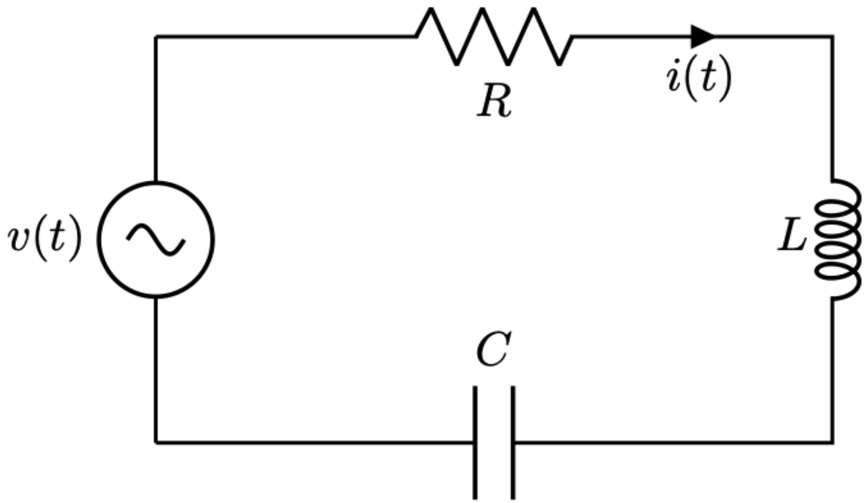
$$\sum_{k=1}^{n} i_k = 0$$

Kirchhoff's Voltage Law (KVL) states that the sum of voltages over m generic branches that form a closed loop (or mesh) is zero.

$$\sum_{k=1}^{m} v_k = 0$$

Single phase AC circuits

Voltages and currents



Circuits where ouput voltages and current are linear combination of input voltages and currents, are, by definition, linear.

In a linear circuit, a sinusoidal current $i(t)$ corresponds to a sinusoidal voltage $v(t)$ **at the same frequency** with a different phase:

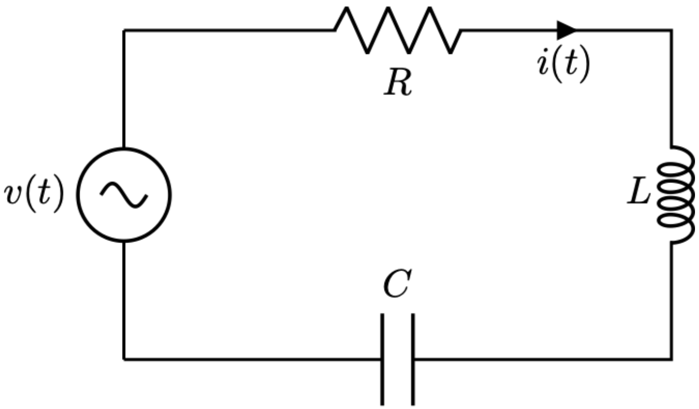
$$v(t) = \sqrt{2}\,V\cos(\omega t + \theta_V) = \Re\mathrm{e}(\sqrt{2}\,\overline{V}e^{j\omega t}) \quad \text{with } \overline{V} = Ve^{j\theta_V}$$

$$i(t) = \sqrt{2}\,I\cos(\omega t + \theta_I) = \Re\mathrm{e}(\sqrt{2}\,\overline{I}e^{j\omega t}) \quad \text{with } \overline{I} = Ie^{j\theta_I}$$

Single phase AC circuits

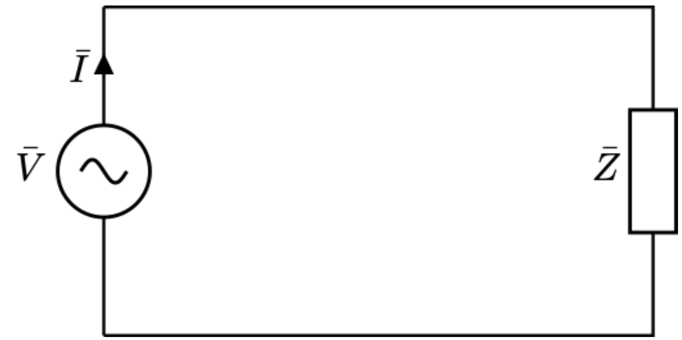
Generic impedance

The equation linking voltage and current across a linear circuit composed by series resistances, inductances and capacitances in the time domain is:



$$v(t) = L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau + Ri(t)$$

In the frequency domain, the equation becomes:



$$\overline{V} = \left[R + j\left(\omega L - \frac{1}{\omega C}\right)\right]\overline{I} = \overline{Z}\,\overline{I}$$

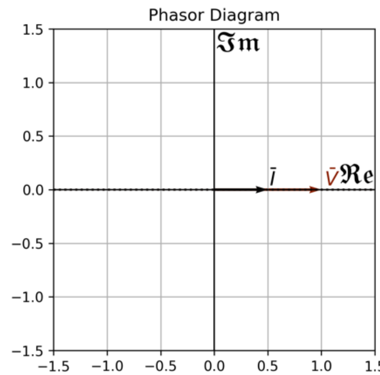
Where $\overline{Z}$ is the impedance (complex number). The latter equation is a simple algebraic equation, easy to solve.

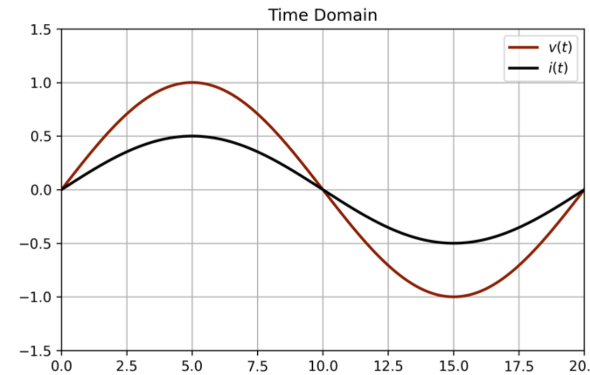
Single phase AC circuits

Resistor, inductor and capacitor

Resistor $\overline{Z} = R$

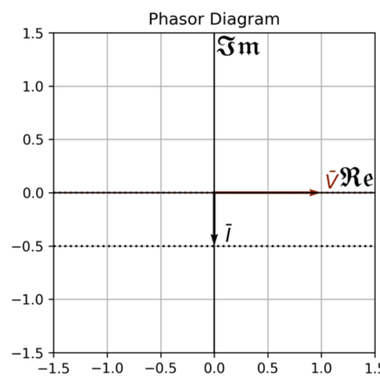
- $v(t)$ and $i(t)$ are in phase
- $\overline{Z}$ is a real number
- $\varphi = 0$

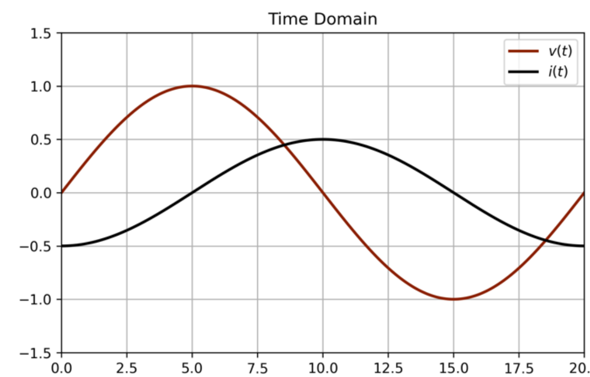




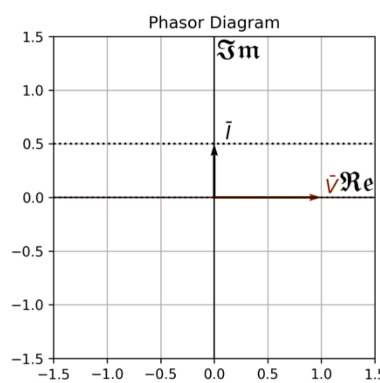
Inductor $\overline{Z} = j\omega L$

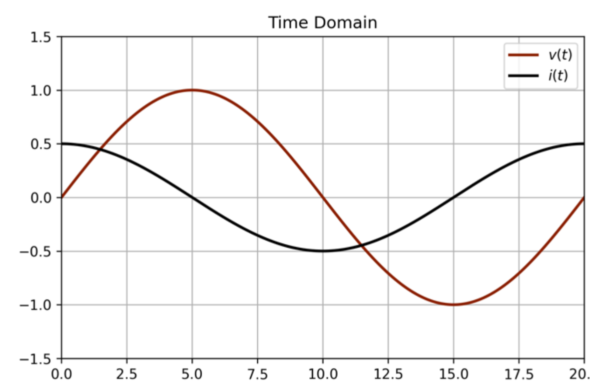
- $v(t)$ leads $i(t)$ by $\pi/2$
- $\overline{Z}$ is an imaginary number
- $\varphi = \pi/2$





- **Capacitor** $\overline{Z} = -j\dfrac{1}{\omega C}$
  - $v(t)$ lags $i(t)$ by $\pi/2$
  - $\overline{Z}$ is an imaginary number
  - $\varphi = -\pi/2$

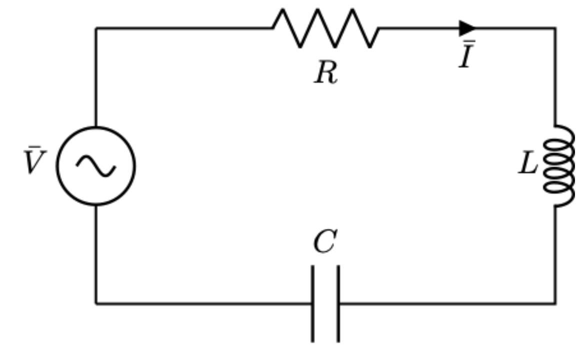




Single phase AC circuits

Generic impedance

$$\overline{Z} = R + j\left(\omega L - \frac{1}{\omega C}\right) = R + jX$$

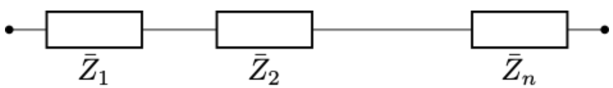


- R is the resistance $[\Omega]$
- X the reactance $X = X_L - X_C = \omega L - \frac{1}{\omega C}$

The reciprocal of impedance is the admittance: $\overline{Y} = \frac{1}{\overline{Z}}$

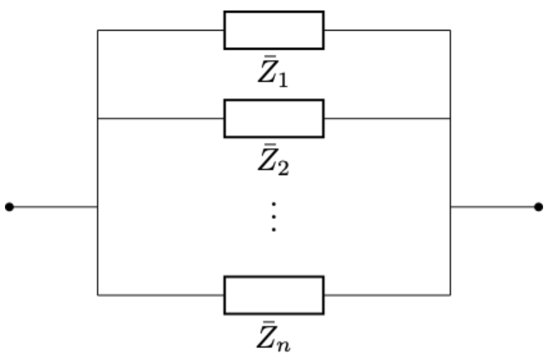
Series of impedances:

$$\overline{Z}_{eq} = \sum_k \overline{Z}_k$$



Parallel of impedances:

$$\frac{1}{\overline{Z}_{eq}} = \sum_k \frac{1}{\overline{Z}_k}$$



Single phase AC circuits

Generic impedance

$$v(t) = \sqrt{2}V\cos(\omega t + \theta_V) = \Re\mathfrak{e}\left(\sqrt{2}\,\overline{V}e^{j\omega t}\right) \to \overline{V} = Ve^{j\theta_V}$$
$$i(t) = \sqrt{2}I\cos(\omega t + \theta_I) = \Re\mathfrak{e}\left(\sqrt{2}\,\overline{I}e^{j\omega t}\right) \to \overline{I} = Ie^{j\theta_I}$$

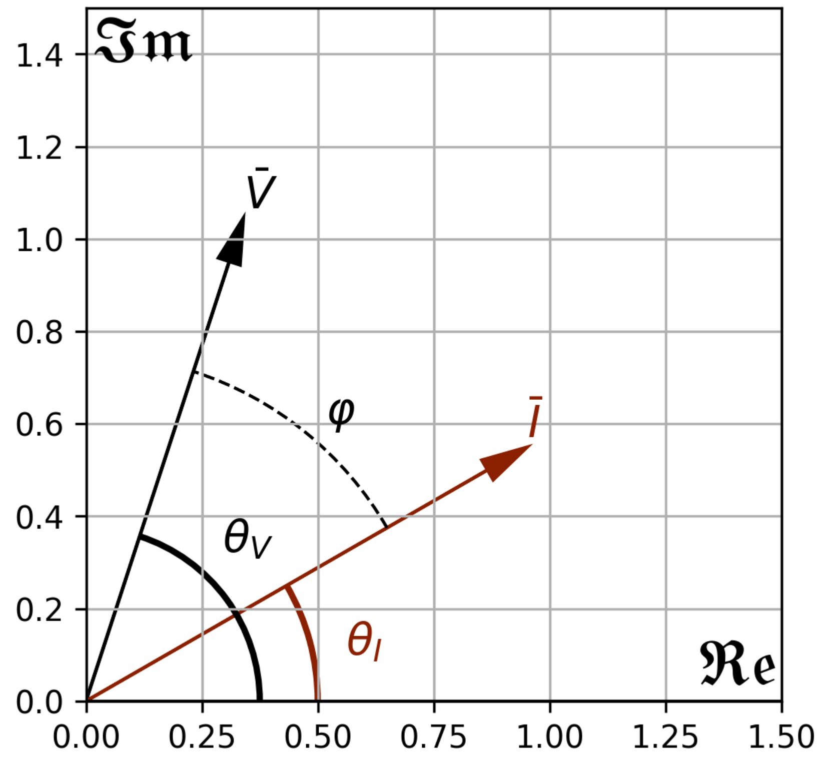
From the definition of impedance:

$$\frac{\overline{V}}{\overline{I}} = \frac{V}{I}e^{j(\theta_V-\theta_I)} = \frac{V}{I}e^{j\varphi} = \overline{Z}$$

Therefore the **phase shift angle:**

$$\varphi = \theta_V - \theta_I$$

between current and voltage phasors is also the **angle of the load impedance**.



Single phase AC circuits

Circuit analysis

Circuit analysis in time domain

KCL

+

KVL

+

$$v_r = f_t(i_r)$$

$\forall$ element r

f_t is an integro-differential equation

Time to phasor transform

KCL

+

KVL

+

$$\bar{V}_r = f_p(\bar{I}_r)$$

$\forall$ element r

f_p is an algebraic equation

Solution

and
Inverse transformation
from the
phasor domain
to the
time domain

Outline

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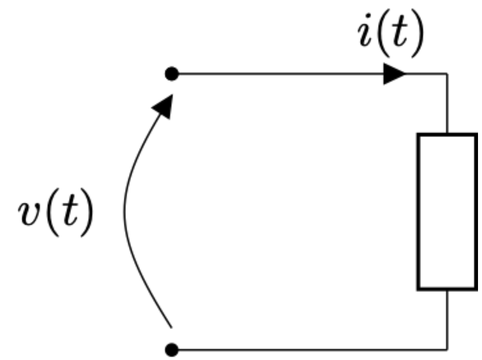
Powers in AC circuits

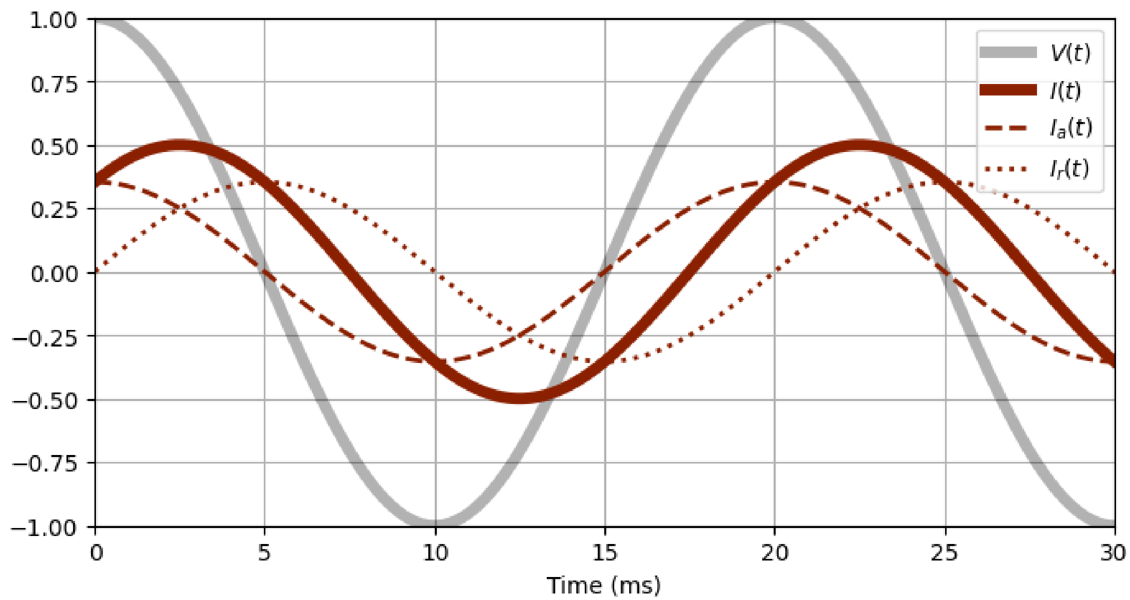
Decomposition of current with respect to voltage



$v(t) = V_{max}\cos(\omega t)$ and $i(t) = I_{max}\cos(\omega t - \varphi)$

$i(t)$ can be transformed as:

$$i(t) = I_{max}(\cos\omega t\,\cos\varphi + \sin\omega t\,\sin\varphi) =$$
$$= I_{max}\cos\varphi\,\cos\omega t + I_{max}\sin\varphi\,\sin\omega t = i_a(t) + i_r(t)$$



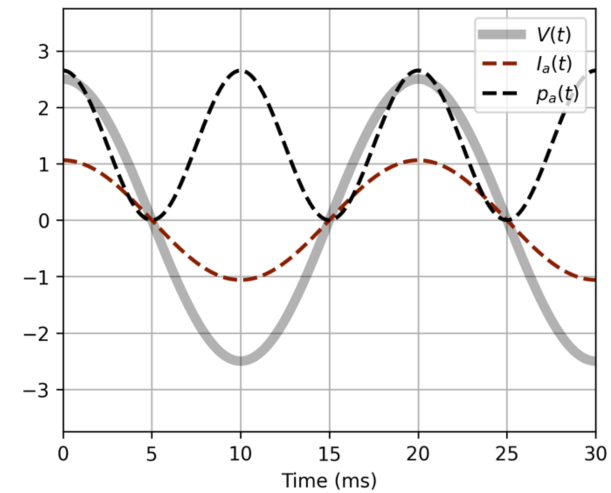
$i_a(t)$ is the **current component in phase with voltage**, and $i_r(t)$ the component **out of phase with voltage**.

Powers in AC circuits

Instantaneous power

The **instantaneous power** is defined as the product of the istantaneous voltage and current:

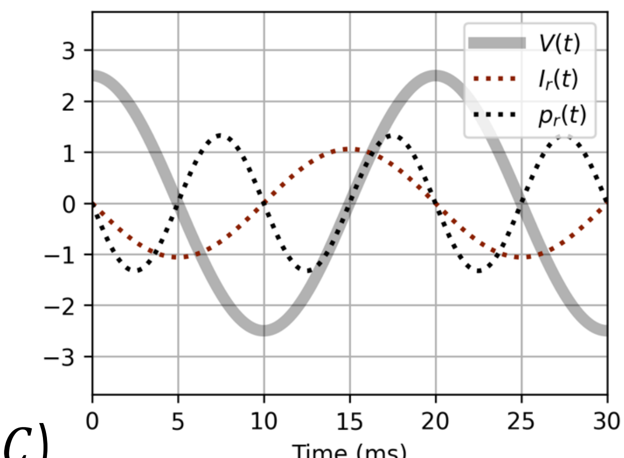
$$p(t) = v(t)i(t)$$



And it is easy to show that it is the sum of:

- the **instantaneous in phase power** $p_a(t)$
- the **instantaneous reactive power** $p_r(t)$

$$p(t) = v(t)i(t) = v(t)i_a(t) + v(t)i_r(t)$$
$$= p_a(t) + p_r(t)$$



Observations:

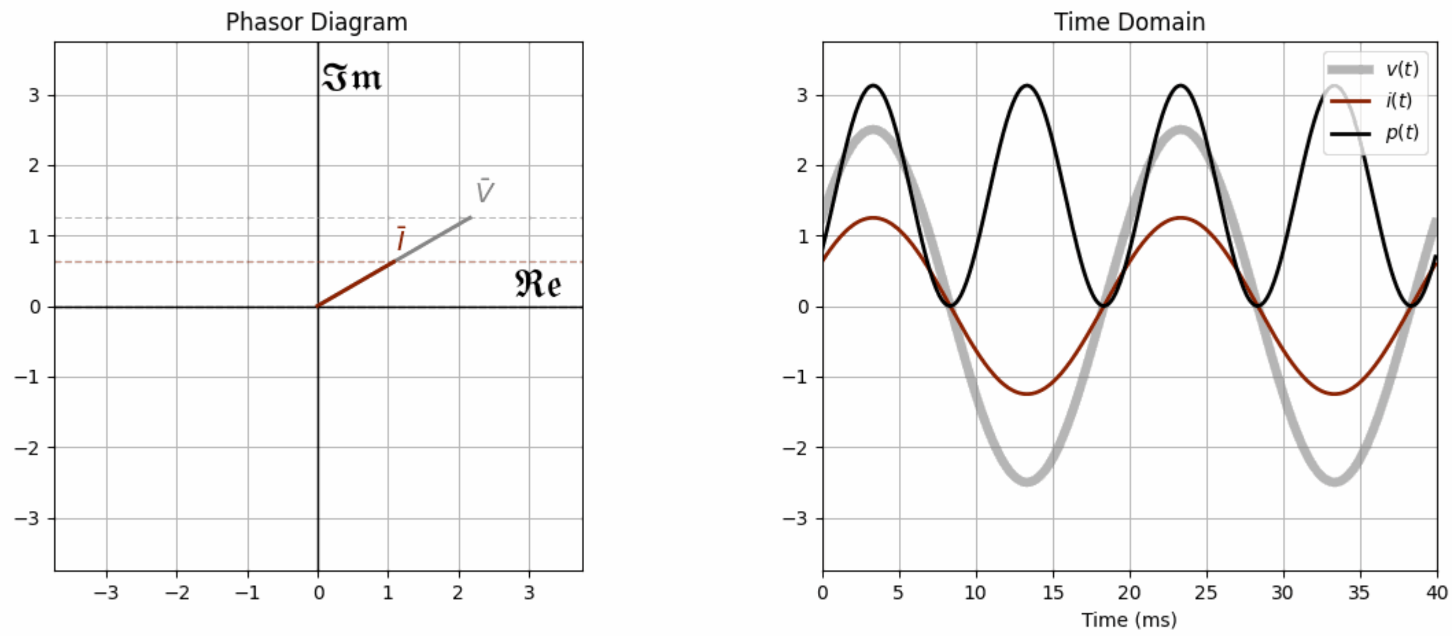
- The average value $\text{avg}(p_a(t)) \neq 0$.
- $p_r(t) \neq 0$ if $\varphi \neq 0$, namely if $L, C \neq 0$.
- The average value $\text{avg}(p_r(t)) = 0$ **→there is energy flowing into the circuit element** (L or C) **for half period and outside of it for the next half period.**

Powers in AC circuits

Instantaneous power

Let's visualise the **instantaneous power** and recall that it is the sum of:
- the instantaneous in phase power $p_a(t)$
- the instantaneous reactive power $p_r(t)$.

$$p(t) = v(t)\ i(t) = v(t)\ i_a(t) + v(t)\ i_r(t) = p_a(t) + p_r(t)$$



Powers in AC circuits

Real (or active) power

The average power P, also known as **real power,** is the average of the instantaneous power over one period T.

$$P = \frac{1}{T}\int_{t_0}^{t_0+T} p(\tau)d\tau = \frac{1}{T}\int_{t_0}^{t_0+T} [p_a(\tau) + p_r(\tau)]d\tau = \frac{1}{T}\int_{t_0}^{t_0+T} p_a(\tau)d\tau$$

Since $p_a(t) = V_{max}\cos\omega\mathrm{t}\, I_{max}\cos\varphi\cos\omega\mathrm{t} = V_{max} I_{max}\cos\varphi\cos^2\omega t$, we get:

$$P = \frac{1}{2} V_{max} I_{max}\cos\varphi = VI\cos\varphi$$

where V and I are the **RMS** values of the voltage and the current, respectively. In the SI System of Units real power is measured in **Watt** [W].

Powers in AC circuits

Reactive power

The **reactive power** Q is the maximum value of the instantaneous reactive power $p_r(t)$:

$$Q = \max[p_r(t)] \cdot sign(\varphi) = \max[V_{max} \cos\omega t \ \ I_{max} \sin\varphi \sin\omega t] \cdot sign(\varphi) =$$
$$= \max\left[V_{max} I_{max} \sin\varphi \ \frac{\sin(2\omega t)}{2}\right] \cdot sign(\varphi) \ = \frac{1}{2} V_{max} I_{max} \sin\varphi = VI\sin\varphi$$

Q is the maximum value of the power exchanged by an inductive or capacitive circuit element with the circuit sources (or with the network to which the element is connected).

Q can be positive or negative depending on the sign of φ. For an inductive load, Q is assumed to be positive by convention and for a capacitive load Q is assumed to be negative.

In the SI system, Q is measured in ***volt - ampere reactive*** [VAr].

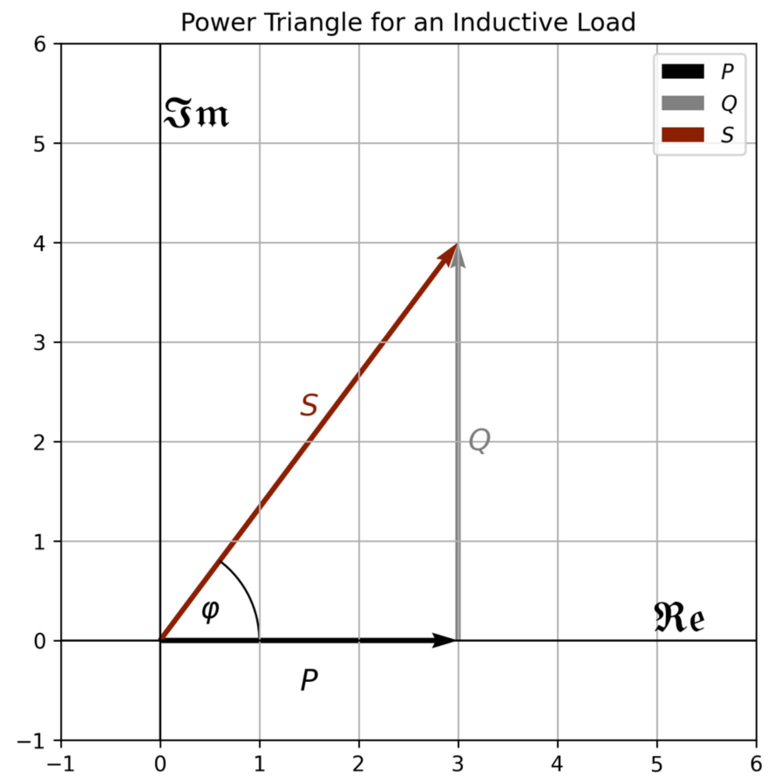
Powers in AC circuits

The power triangle

$$\bar{S} = \bar{V}\,\bar{I}^* = VIe^{j\varphi} = VI(\cos\varphi + j\sin\varphi) = P + jQ$$

$$P = Re(\bar{S}) \quad ,Q = Im(\bar{S})$$

$\bar{S}$ is called is **apparent power** and is a complex number.



The **power factor** is the cosine of the phase difference between voltage and current. Hence, it is the cosine of the angle of the load impedance:

$$P = VI\cos\varphi = |\bar{S}|\cos\varphi$$

$$\cos\varphi = \frac{P}{VI} = \frac{P}{|\bar{S}|}$$

Trivially, the value of the power factor $\cos\varphi$ ranges between 0 and 1.

Powers in AC circuits

Power factor

- For a **purely resistive load**, the voltage and current are in phase, i.e., $\varphi = \theta_V - \theta_I = 0$ and the power factor $\cos\varphi = 1$. Therefore the apparent power is equal to the real (or active) power.

- For a purely reactive load $\varphi = \theta_V - \theta_I = \pm\pi/2$ and the $\cos\varphi = 0$. In this case the real power is zero.

- In between these two extreme cases the power factor is said to be leading or lagging. Leading power factor means that the current leads the voltage **(i.e., the load is capacitive)**. Lagging power factor means that the current lags the voltage **(i.e., the load is inductive)**.

Powers in AC circuits

Power factor

$$P = VI\cos\varphi = S\cos\varphi \Leftrightarrow \cos\varphi = \frac{P}{VI} = \frac{P}{S}\frac{Q}{P} = \frac{VI\sin\varphi}{VI\cos\varphi} = \tan\varphi \Leftrightarrow \cos\varphi$$
$$= \cos\left(\tan^{-1}\frac{Q}{P}\right)$$
$$I = \frac{P}{V\cos\varphi}$$

At fixed P and V if $\cos\varphi \downarrow$, then $I \uparrow$.

In general, the power factor of loads has to be as close as possible to 1 to reduce the magnitude of current supplying the loads (that produces power losses into lines).
Some utilities request to pay the power factor utilized when its value is below 0.9.

Outline

Introduction

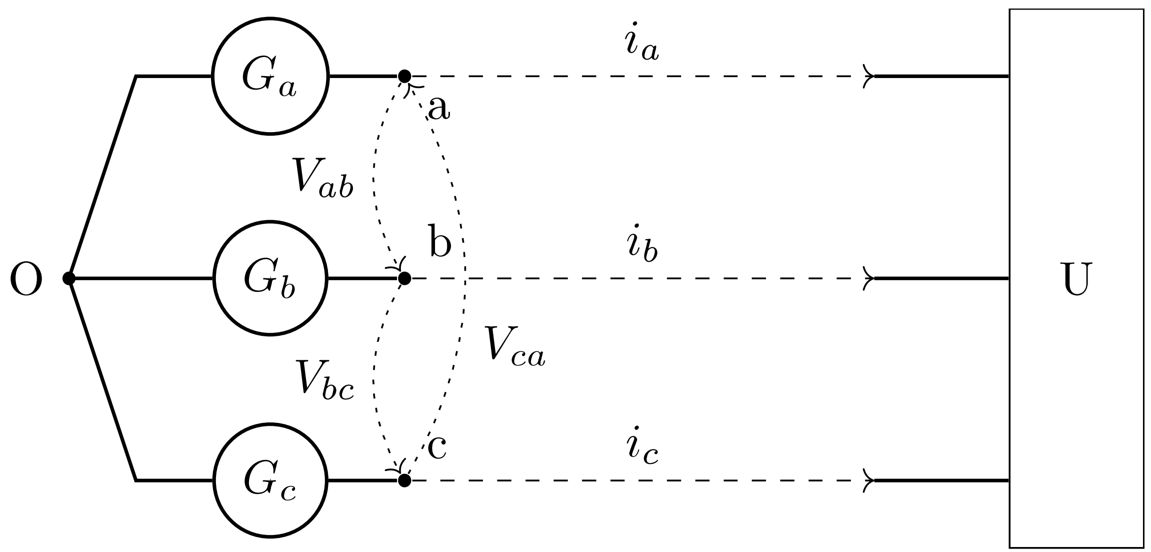
Single-phase AC circuits

Powers in AC circuits

Three-phase AC circuits

Three phase AC circuits

Voltages and currents in balanced+symmetrical 3-ph systems

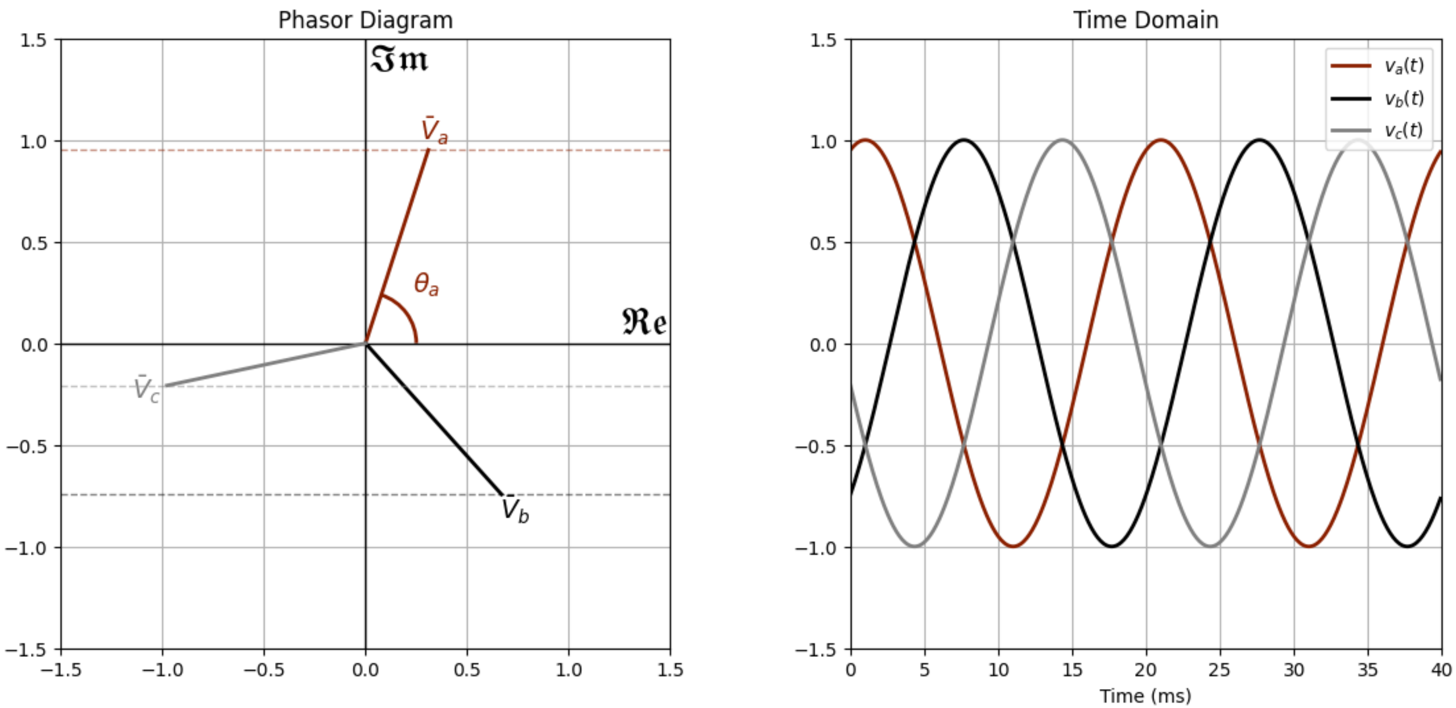


The generation and the distribution of electrical energy is usually done by **three-phase systems**. There are three wire systems connected to a generator consisting of three AC sources having the same amplitude and frequency (mostly $50\ Hz$ in Europe as well as most of Asia and Australia, and $60\ Hz$ in North America and Canada) but shifted in phase by $\frac{2}{3}\pi$ (i.e., $120deg$).

Three phase AC circuits

Motivations

1. It is easy to convert mechanical into electrical power and vice versa, using rotating three phase machines.
2. For the same amount of transported power, a three phase line uses less conductive material to build a corresponding single phase line.
3. In 3-ph systems, the instantaneous power is constant, resulting in a uniform transmission and less vibrations.



Three phase AC circuits

Balanced and symmetrical 3-ph systems

Balanced System: in a balanced system, the sum of the three phasors of currents or voltages is zero.

Symmetrical Systems: in a symmetrical system, the angles between subsequent phasors of voltages or currents are equal.

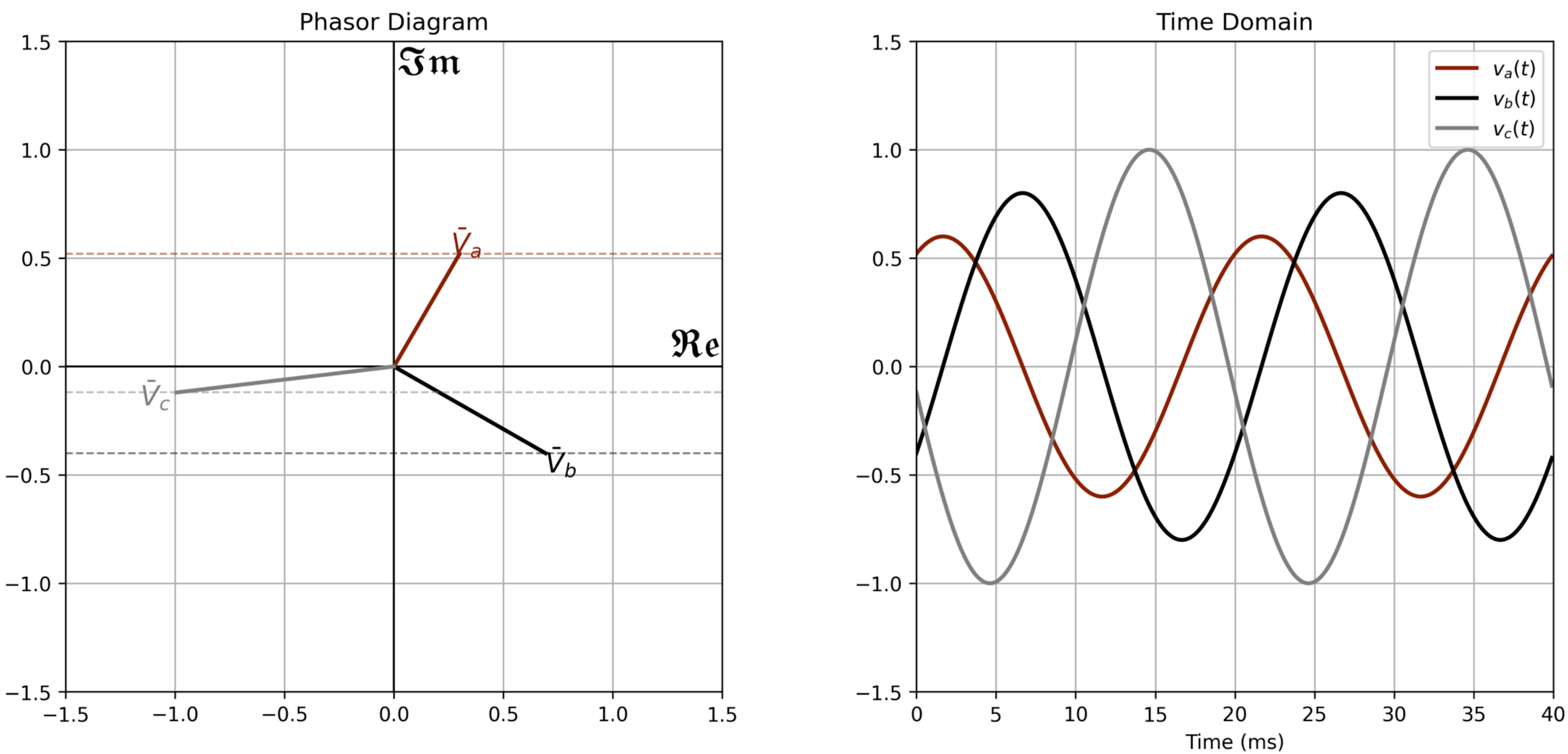
Important:

1. A **balanced system** is **not necessarily symmetrical**.
2. A **symmetrical system** is **not necessarily balanced**.
3. In a **balanced and symmetrical 3 phase system, the phases between sub-sequent phasors of voltages and currents are equal to** $\frac{2}{3}\pi$ **and their magnitudes are identical.**

Three phase AC circuits

Balanced and symmetrical 3-ph systems

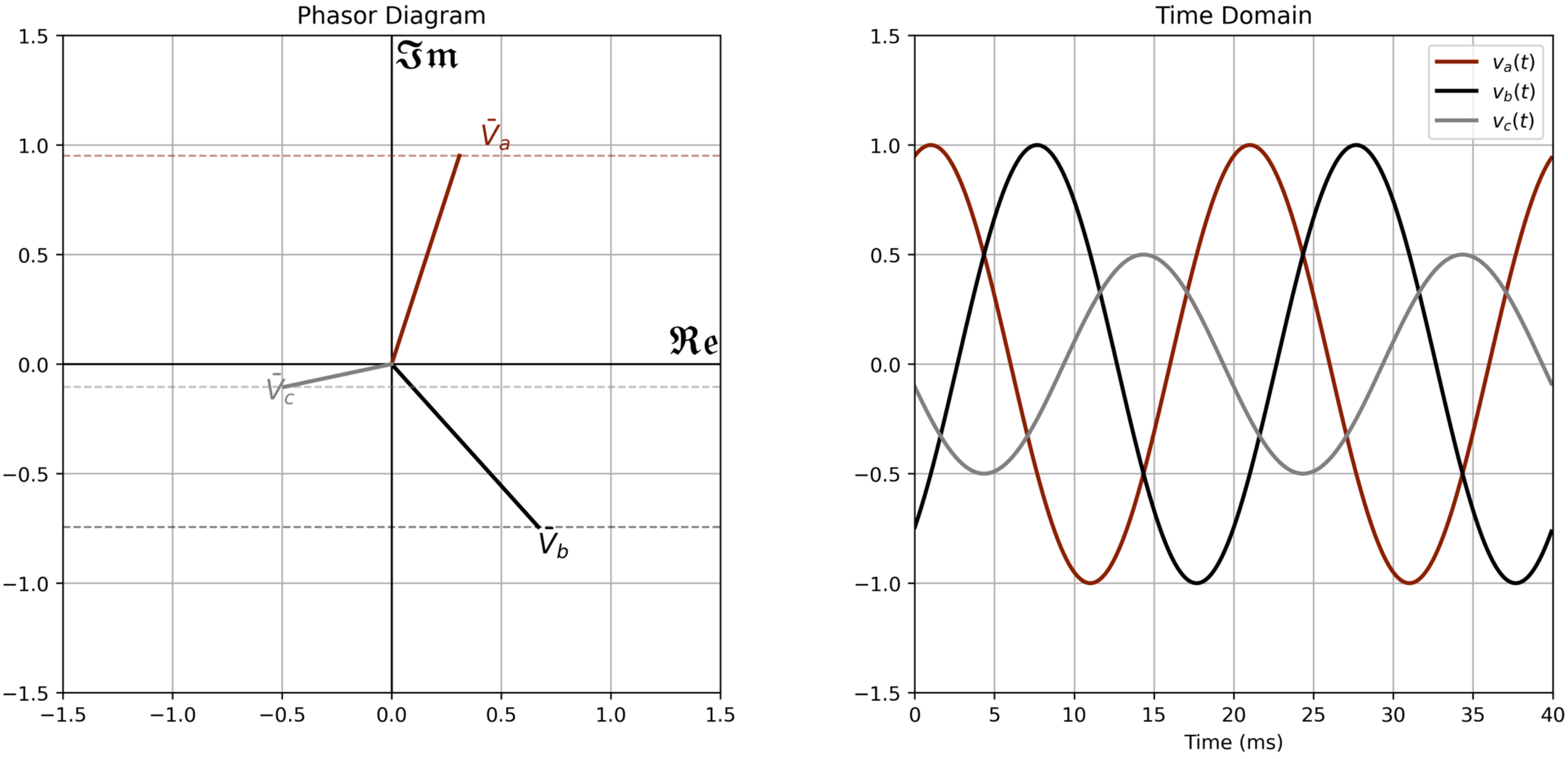
1. **A balanced system is not necessarily symmetrical.**



Three phase AC circuits

Balanced and symmetrical 3-ph systems

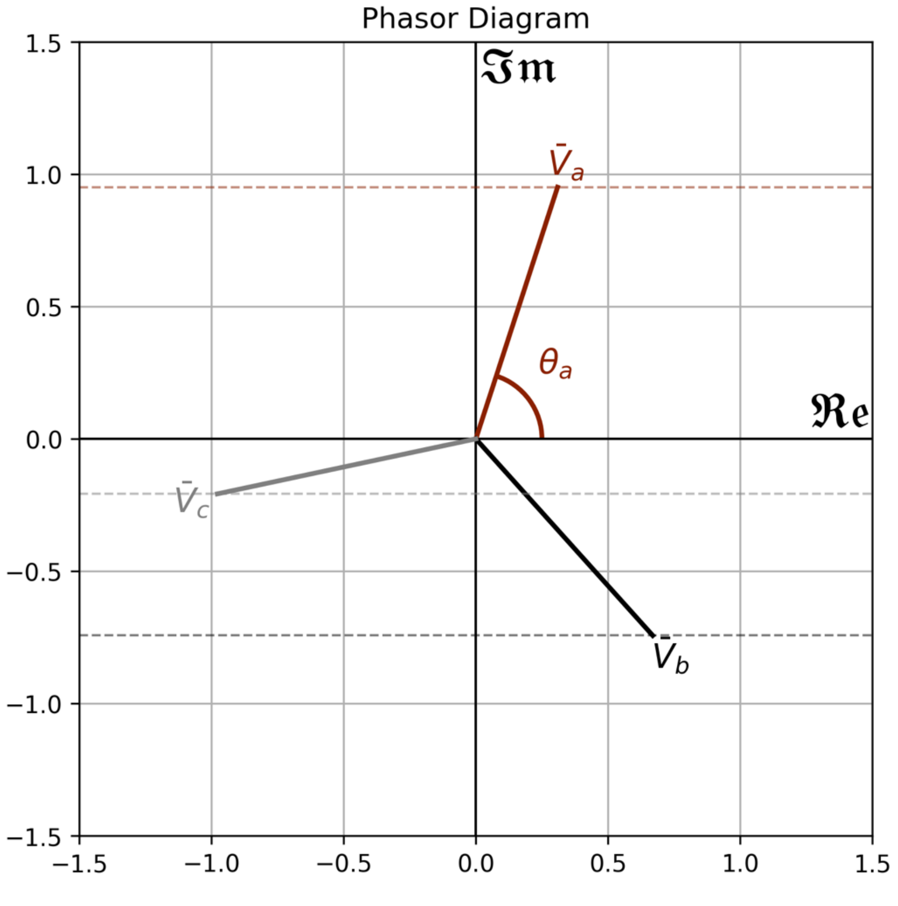
2. A symmetrical system is not necessarily balanced.

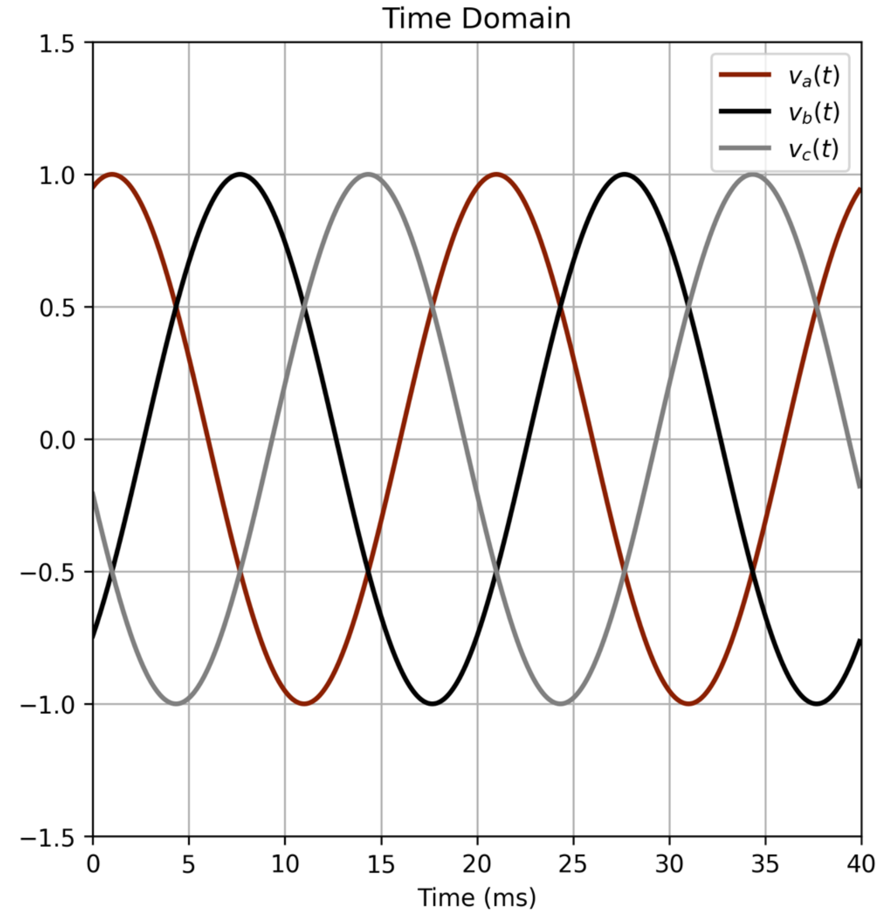


Three phase AC circuits

Balanced and symmetrical 3-ph systems

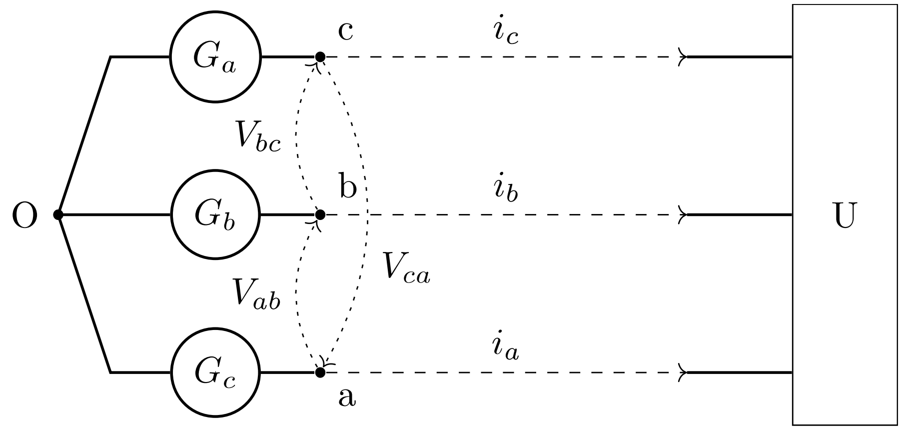
3. In a balanced and symmetrical 3-ph system the phases have a precise 120-degree phase separation.





Three phase AC circuits

Voltages and currents in balanced 3-ph systems



The line currents $i_a(t), i_b(t)$, and $i_c(t)$ are the currents flowing in each of the three phases. For the system in the figure, by applying the **KCL**, we get:

$$i_a(t) + i_b(t) + i_c(t) = 0$$

The **phase-to-phase** (or **line-to-line**) voltages $v_{ab}(t), v_{bc}(t), v_{ca}(t)$ are the voltage differences between terminals ab, bc and ca. For the system in the figure, by applying the **KVL**, we get:

$$v_{ab}(t) + v_{bc}(t) + v_{ca}(t) = 0$$

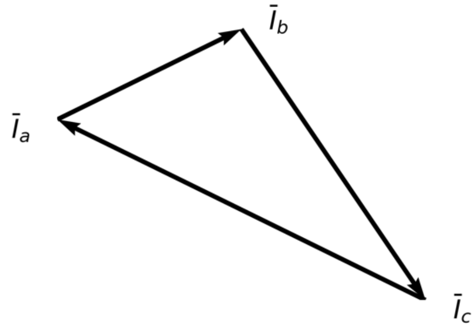
Three phase AC circuits

Voltages and currents in balanced 3-ph systems

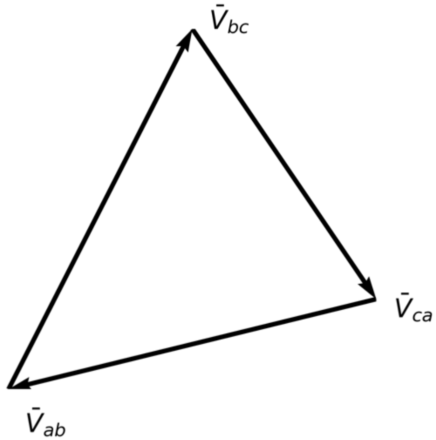
We assume that the three-phase system is **iso-frequency** (i.e., the three phase voltages and currents have the same frequency). Therefore, the **KCL and KVL written before can be also written in the phasor domain:**

$$\bar{I}_a + \bar{I}_b + \bar{I}_c = 0, \quad \bar{V}_{ab} + \bar{V}_{bc} + \bar{V}_{ca} = 0$$

As a consequence, the **three phase line currents** can be represented by the **triangle of the line currents** and the **three phase-to-phase voltages by the triangle of voltages**:



Triangle of line currents



Triangle of phase-to-phase voltages

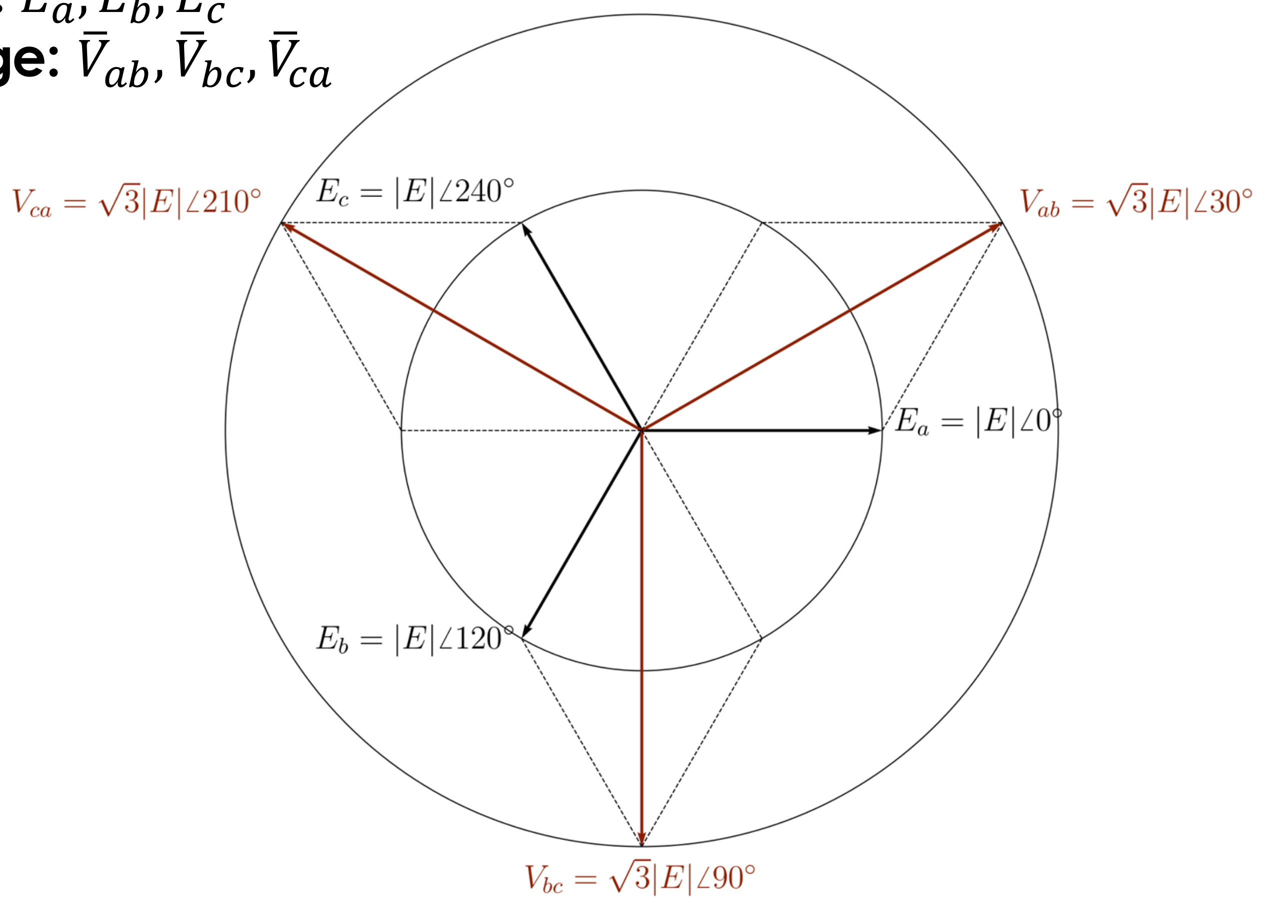
Three phase AC circuits

Voltages and currents in balanced and symmetrical 3-ph systems

The single-phase voltage $\bar{E}_a, \bar{E}_b, \bar{E}_c$ are the voltage difference between the center O (see figure before) and the terminals a, b, c.
Single phase voltages: $\bar{E}_a, \bar{E}_b, \bar{E}_c$
Phase-to-phase voltage: $\bar{V}_{ab}, \bar{V}_{bc}, \bar{V}_{ca}$

We also have that:

$$\overline{V}_{ab} = \overline{E}_a - \overline{E}_b$$
$$\overline{V}_{bc} = \overline{E}_b - \overline{E}_c$$
$$\overline{V}_{ca} = \overline{E}_c - \overline{E}_a$$



Three phase AC circuits

Voltages and currents in balanced and symmetrical 3-ph systems

The **total real** and **reactive powers** in a three phase balanced and symmetrical system where E and V are the RMS values of the single phase and phase-to-phase voltages, respectively, is given by:

$$P = 3EI\cos\varphi = \sqrt{3}VI\cos\varphi$$

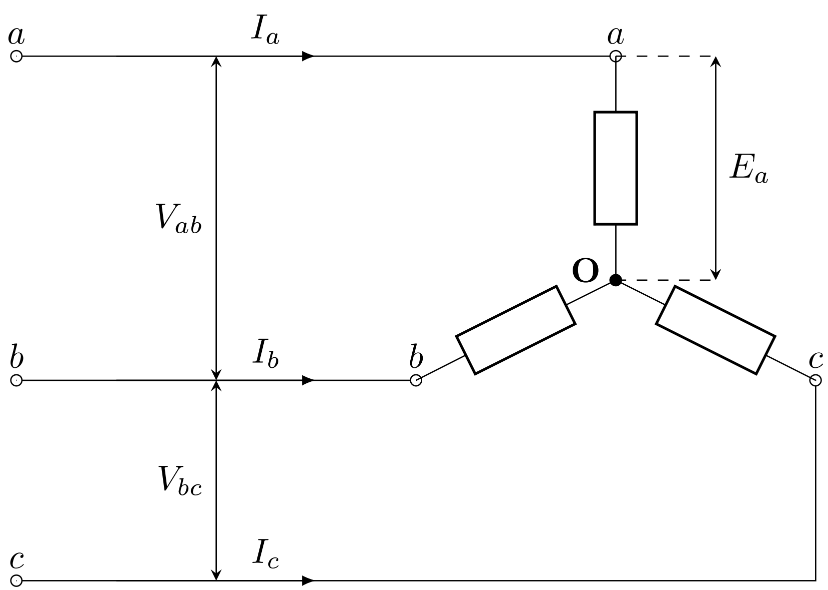
$$Q = 3EI\sin\varphi = \sqrt{3}VI\sin\varphi$$

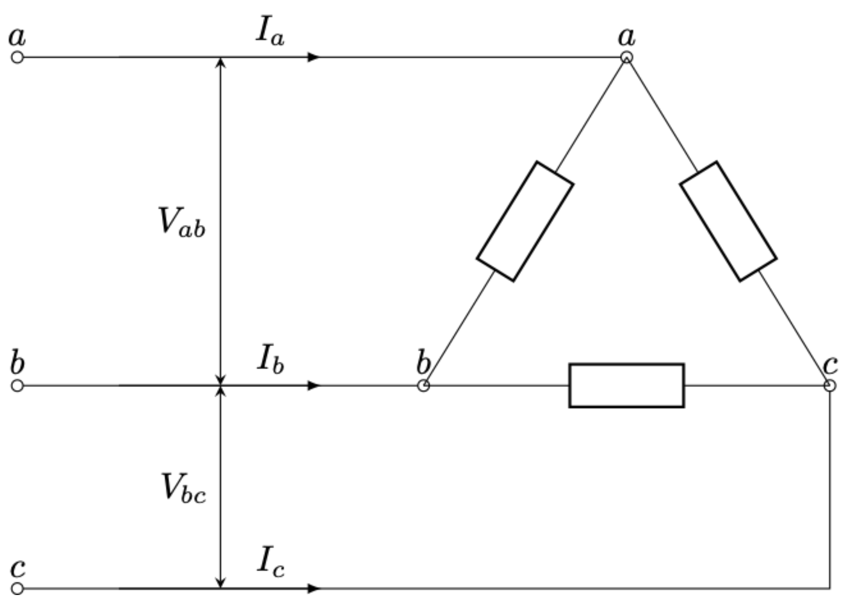
The power factor is:

$$\cos\phi = \cos\left(\tan^{-1}\left(\frac{Q}{P}\right)\right)$$

Three phase AC circuits

Y and Δ connections





Connection	Voltage Across Impedance	Current Across Impedance
Star	$E = V/\sqrt{3}$	I
Delta	$V = \sqrt{3}E$	$I/\sqrt{3}$